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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) geometry and experience; (2) geometry and empirical science; (3) physical geometry; (4) dimension; and (5) curves. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

Mathematics has often been characterized as the "Handmaiden of the Sciences." It would seem that somehow, mysteriously, mathematics is a powerful tool which "explains" or "describes" Nature in a remarkably convenient and uncannily accurate way. The curious observer may well wonder how to account for this apparently fortuitous piece of serendipity. Einstein himself once raised that very question: "How can it be that mathematics, being after all a product of human thought independent of experience, is so admirably adapted to the objects of reality?" Elsewhere[1] he gave a partial answer to his own question: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

A more explicit answer may perhaps be found in the way in which mathematics use *causal models*. We cannot *know* the "laws" (if any) that govern our physical environment; we can only observe the *effects* of these laws. Thus, out of sheer human curiosity, when confronted with a physical situation, we idealize the circumstances and create a hypothetical model as nearly like the physical situation; then we attempt to study the model mathematically. The general procedure is suggested by Swann,[2] as follows:

> The pure mathematician . . . will set up a branch of mathematics founded upon certain postulates having to do with quantities, letters, etc., that he chooses to be talking about. In this mathematical scheme, there will appear relationships between certain quantities which occur in the mathematics, and it will be his hope to invent a scheme of mathematics of this kind which shall form an analog of the regularities of nature in the sense that there may be a one-to-one correspondence between certain things in the mathematics and the observable phenomena in nature. . . . When the correspondence has been set up, the postulates of his mathematics become the laws of nature in the physics.

Or, in the cogent words of another celebrated physicist:[3]

> "On the one hand, mathematics is a study of certain aspects of the human thinking process; on the other hand, when we make ourselves

[1] Albert Einstein: *Geometrie und Erfahrung.*

[2] W. F. G. Swann, "Reality in Physics," *Science* 75:113–114 (1932).

[3] W. Heisenberg, as quoted by P. W. Bridgman in *Science*, 1930, vol. 71, p. 21.

master of a physical situation, we so arrange the data as to conform to the demands of our thinking process. It would seem probable, therefore, that merely in arranging the subject in a form suitable for discussion we have already introduced the mathematics — the mathematics is unavoidably introduced by our treatment, and it is inevitable that mathematical principles appear to rule nature."

These passages both give plausibility to a viewpoint expressed a few years earlier by J. W. N. Sullivan,[4] a perceptive philosophical observer of science, in which he suggested that

> " . . . the significance of mathematics lies precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends upon our minds. It does not enable us to explore some remote region of the externally existent; it helps to show us how far what exists depends upon the way we exist. We are the lawgivers of the Universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the material universe itself."

Enough has been said here to indicate that the relation of mathematics to physical science is far from a simple matter. The present group of essays will make these concepts more meaningful, especially as they concern the relation of geometry to empirical science and measurement.

— William L. Schaaf

[4] J. W. N. Sullivan, *Aspects of Science, Second Series*, Alfred Knopf, 1926, pp. 93 ff.

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FOREWORD

No doubt you have at one time or another come across so-called optical illusions, such as a drawing in which one of two equal line segments appears at first glance to be definitely longer than the other; or a drawing in which parallel lines seem to bend, or where an apparently curved line is actually made up of many small straight line segments.

How do we come to recognize and identify such matters as length or direction, straight or curved, flat or round, and so on? It is questions like these that are of interest to the physiologist, the engineer, the architect, the designer, and the mathematician.

For although geometry as a mathematical discipline is not in the least concerned with material or physical objects, nevertheless many of the basic concepts of geometry are suggested by human experience when observing and handling material objects. Not only the objects themselves, but the ways in which they are orientated to one another also suggest geometric concepts such as horizontal, vertical, oblique, parallel, perpendicular, symmetric, congruent, and the like. In the present article the author explores the sources of our imagery and our conception of space and spatial relationships. As such, the essay furnishes a most appropriate introduction to the succeeding paper on the relation of geometry to empirical science, which is somewhat more sophisticated and rather profound.

GEOMETRY AND EXPERIENCE*

N. A. COURT

STUDENTS who gather for their first lesson in geometry know already a good deal about the subject. They are familiar with certain shapes that textbooks on geometry call parallelepipeds, spheres, circles, cylinders, which the students would call boxes, balls, wheels, pipes. Notions such as point, line, distance, direction, and right angle are quite familiar and clear to them, in spite of all the difficulties learned mathematicians profess to encounter when they try to clarify or define these concepts.

The question arises, how was this store of knowledge gathered, how was this information acquired? The empiricists maintain that geometrical knowledge is the result of the experience of the individual in the world surrounding him. However, the universal acceptance of the basic properties of space lead the apriorists to the conclusion that these spatial relations are innate, that they constitute a fundamental characteristic or limitation of the mind which cannot function without it or outside of it. The invention of non-Euclidean geometry has done considerable damage to the solidity of the apriorist armor but has not eliminated the debate between the two schools of thought.

During the present century the eminent French sociologist Émile Durkheim (1858–1917) advanced an intermediate thesis. The source of our geometric knowledge is experience. However, at a very early stage of civilization this individual experience is pooled and codified by the group, owing to social necessity and in order to serve social purposes. Our basic geometric knowledge is thus a social institution. It is this social function of geometry that accounts for the fact of its universal acceptance, for the inability of the individual to act contrary to it, for the mind to reject it.

It is universally agreed that the actual experience of living is the basic factor in the process of accumulating information of the kind that we call spatial or geometrical. This in turn amounts to saying that we come into possession of this information through our senses. Such being the case, the question naturally comes to mind, which of our senses is it that performs this function?

*From an address before the Mathematical Colloquim, University of Oklahoma, May 1944.

The sense of hearing helps to acquire the notion of direction. To a lesser degree this is also true of the sense of smell. The sense of taste need hardly be mentioned in this connection. The sense of sight and the sense of touch remain. It does not take much effort to see that these two senses play the dominant part in the shaping of our geometrical knowledge.

The sense of touch, considered in its broader aspect of including also our muscular sense, supplies us with information as to the shape of things. It is also our first source of information about distance. By touch we learn to distinguish between round things and things that have edges, things that are flat and things that are not flat. It is the sense of touch that conveys to us the first notions of size. This object we can grasp with our hand, and this other cannot be so grasped; it is too big; this object we can surround with our arms, this other we cannot; it is too big.

These examples imply measuring and the measuring stick is the size of our hand, the length of our arm, and, more generally, the size of our body. The whole environment that we have created for ourselves in our daily life is made to measure for the size of our body. That the clothes we wear are adapted to the size of our body and our limbs goes without saying. But so is the chair we sit on, the bed we sleep in, the rooms and the houses we live in, the steps we climb, the size of the pencil we use, and so on without end. We take it so much for granted that things should fit our size that we are startled when they fail to conform to the adult standard, as, for example, in the children's room of a public library where the chairs are tiny and the tables very, very low. The legendary robber Procrustes, of ancient Greece, had his own ideas about matching the sleeper and the size of the bed. He made his victims occupy an iron bed. If the occupant was too short, he was subjected to stretching until he reached the proper length. If, on the contrary, the helpless victim was too tall he was trimmed down to the right size, at one end or the other. Hebrew writers placed this famous bed in Sodom, and it was one of the iniquities that caused Sodom's destruction, by a "bombardment from the air."

In many cases the fact that things are made on the "human scale" may be less immediate but is no less real. The clock on the wall has two hands, whereas, strictly speaking, the hour hand alone should be sufficient. Owing to the limitations of our eyesight, we cannot evaluate with sufficient accuracy fractional parts of an hour by the use of the hour hand alone, unless the face of the clock was made many times larger than is customary. But then the clock would become an unwieldy object, out of proportion to the other objects around us made to the "human scale."

The comparison of the size of objects surrounding us with the size

of our body is not just a kind of automatic reflex but is a deliberate operation as well. When in the course of our cultural development the need arose for greater precision in describing sizes and for agreement upon some units of length, we turned to our body to provide the models. The length of the arms and of the fingers, the width of the hand, the length of the body and of the legs all served that purpose at one time or another, at one place or another. The yard is, according to tradition, the length of the arm of King Henry I. The origin of the "foot" measure requires no explanation, and we still "step off" lengths.

The sense of vision is the other great source of geometrical information. To a considerable extent this information overlaps the data furnished by the sense of touch. Sight informs us of the difference in sizes of objects around us. Sight supplements and extends the notion of distance that we gain through touch. Sight tells us of the shape of things, and on a much larger scale than touch does. But sight asserts its supremacy as a source of geometrical knowledge when it comes to the notion of direction. Moreover, sight tells us "at a glance" which object is closer, which is farther, which is in front and which is behind, which is above and which is below. Sight is supreme in telling us when objects are in the same direction from us, when they are in a straight line. When we want to align trees along our streets, we have recourse to sight. The fact that light travels in a straight line is one of the main reasons for the dominant position the straight line occupies in our geometrical constructs. I realize that some learned persons will smile indulgently at the statement that a ray of light is rectilinear. I will, nevertheless, stick to my assertion as far as our terrestrial affairs are concerned, whatever may be true of light on the vaster scale of the interstellar or intergalaxian universe.

Up to this point the geometrical knowledge I have mentioned is the kind familiar to "the man in the street." Let us now turn to the systematic study of the subject, to the science of geometry. Are both empirical sources of geometrical knowledge reflected in systematic geometry? Is it possible to classify geometrical theorems on that basis?

If we examine Euclid, we see that he leaned heavily towards tactile geometry, or the geometry of size. His main preoccupation was to establish the equality of segments and angles, to prove the congruence of triangles. The method of proving triangles to be congruent consists in picking up one triangle and placing it on the top of the other, which implies that the moving triangle does not change while it is in motion. This possibility of rigid motion was much insisted upon by Henri Poincaré (1854–1912) and is now considered by mathematicians to be the characteristic property of the geometry of size, or, to use the professional

fundamental importance, in the collection of Pappus, a Greek author of the third century of our present era. A systematic study of visual geometry had to wait for a millennium and a half before it found its apostle and high priest in the person of the French army officer Jean Victor Poncelet (1788–1867), the father of projective geometry.

Consider any geometrical figure, say a plane figure (triangle) F (Fig. 1), for the sake of simplicity, and let S be a point (representing the eye) not in the plane of figure F. Imagine the lines joining every point of figure F to the point S. Now, if we place a screen between S and figure F, everyone of these lines will mark a point on the screen, and thus we obtain a new figure F′ in the new plane, the image of figure F.

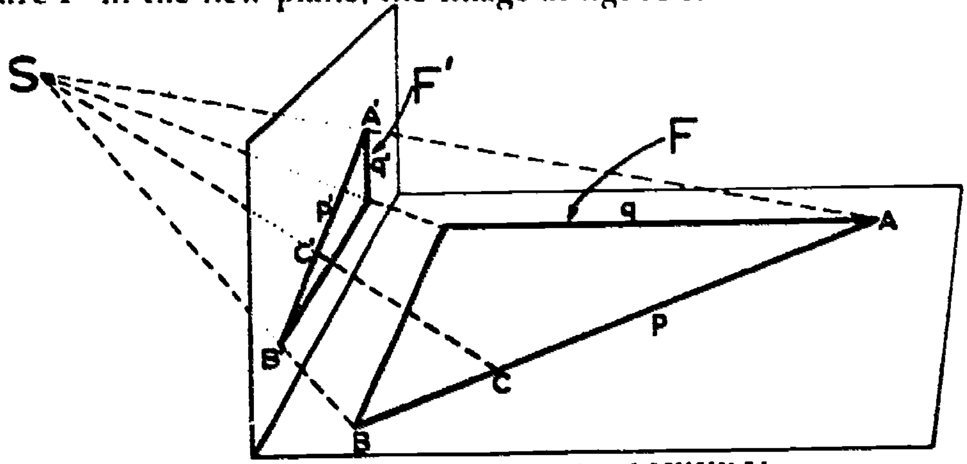


FIG. 1. *IN RE* PROJECTIVE GEOMETRY

If we compare the two figures F and F′, we notice some very interesting things. The figure F′ in general will be different from F. It has suffered many distortions. If A, B are two points in F and A′, B′ are their images in F′, the distance A′B′ is not equal to the distance AB, as a rule, and may be either smaller or greater than AB, and this alone deprives the figure F′ of any value in the study of the figure F from a metrical point of view. There are, moreover, many other distortions of various kinds. But some characteristics of F always reappear in F′. Of these the most important is that a straight line p of F has for its image in F′ a straight line p′, and consequently any three points A, B, C of F that lie on a straight line in F will have for their images in F′ three points A′, B′, C′ that also lie in a straight line. If two lines p and q are taken in F, their images in F′ are two straight lines p′ and q′, but the angle p′q′ is not equal to the angle pq, as a rule, and may be either smaller or larger than pq. In particular, the images of two parallel lines are not necessarily parallel, and the images of two perpendicular lines are not necessarily perpendicular.

If we call figure F′ the projection of figure F from the point S, we may say that projection preserves incidence and collinearity. The systematic study of projective geometry, or visual geometry, is the study of those

properties of figures that remain unaltered by projection, just as it may be said of metrical geometry that it is the study of those properties of figures that remain unaltered in rigid motion.

From the point of view of the theory of knowledge it is of great significance that the distinction between tactile geometry and visual geometry was not noticed by either philosophers or psychologists. Only after the patient labors of mathematicians created the doctrine of projective geometry did the distinction come to light. The credit for having pointed out this distinction goes to Federigo Enriques, Professor of Projective Geometry at the University of Rome.

In the study of the sources of our geometrical knowledge too little attention is accorded to our own mobility, to our ability to change places. Even the range of our knowledge due to touch is considerably increased by our ability to move our arms. In connection with our visual information our mobility is of paramount importance. To mention only one point, the shape of an object depends upon the point of view, or the point of observation. It is our ability to change places that makes it possible for us to eliminate the fortuitous features from our observations.

As has been mentioned before, our tactile and visual information do not cover the same ground, but they overlap to a considerable extent and thus complement each other. But do they always agree? If a person drives his car over a stretch of straight road, he observes that the road is of the same width all along. He knows it to be so by comparison with the size of his car and by comparison of the size of his car with his own size; in other words, it is a tactile fact. Now, if he turns around and looks at the road just traversed, he sees "with his own eyes" that the road is getting narrower as it extends back into the distance and seems to vanish into a point. These two items of information on the same subject contradict each other. Which of them is true and which is false? Which of term, of metric geometry. Euclid's is thus metrical geometry exclusively, or nearly so. This is not at all surprising, since metrical geometry is the geometry of action, the geometry that builds our dwellings and makes our household utensils. The very origin of Euclid's geometry is supposed to be connected with the parcelling out of plots of land in Egypt after the recession of the flood waters of the Nile.

Euclid did not know that his was metrical geometry. To him it was just geometry, for he knew of no other kind. Neither did his successors, in spite of the fact that they added to Euclid's *Elements* a considerable number of geometric propositions which in their nature are visual and not metric. There are numerous such propositions, some of them of

them do we accept and which do we reject? Above all, how do we go about telling which to accept and which to reject?

When one puts a perfectly good spoon into a glass of water, he sees that the spoon is unmistakably broken, or at least bent at a considerable angle. He takes the spoon out, and it is as good as it was before he put it in. He runs his finger along the spoon while it is in the glass and feels that it is as straight as ever. But when he looks at it, there is no doubt that the spoon is bent; contradictory testimony of two different senses. Again the question arises, which of the two pieces of information do we accept, and on what ground do we make our choice?

A long time ago I read of a lake where the water was so clear that on a bright moonlit night it was possible to see the fish asleep on the bottom of the lake. Devotees of fishing would take advantage of this situation and go out in a boat, as quietly as possible, to the middle of the lake and then try to catch the fish by striking them with a harpoon. It was explained in my reader that aiming the harpoon at the spot where a fish was seen would spell disastrous failure and that successful practitioners of the sport would know the spot at which to aim, although the fish was seen to be elsewhere.

The moral of this fish story is of great importance. In the case of the road and in the case of the spoon we all repudiate the testimony of our eyes and accept the verdict of the sense of touch. We do so whenever the tactile and the visual testimonies are in disagreement. But why?

The answer to this puzzling question may be found in the activity of man. Moreover, his activities are purposeful and must be coordinated so as to achieve success. Now, man's organs of activity, his hands, are also the main organs of touch. Man has thus developed a close co-ordination between his touch and his actions. At short range, he has implicit faith that his actions will be fruitful if he relies on the data furnished by touch. Visual data concern objects at a distance and serve well as a first approximation. They are good in most cases but are always subject to control and check. If light sees fit to indulge in such vagaries as reflection, refraction, and mirages, so much the worse for light. My fish story points to just that moral. Sight leads us to the fish. But if we want to act on it successfully, we must subject this information to the necessary correction as learned by touch. Otherwise we shall have no fish to fry.

FOREWORD

As a mathematical discipline, geometry—that is, pure geometry, is without "content" in the sense that it makes no pretense at describing the properties of physical objects or any relations between such objects. It deals instead with "ideals" or ideas; these do not "exist" in the sense that material objects exist for us. Moreover, we make our own "rules of the game" when we explore these ideals. Thus a geometry—any geometry—can never be proved right or wrong, and certainly not by observing or measuring physical objects in support of a particular system of geometry.

Therein lies the essential difference between physical science and geometry. In the case of science, we arrive at generalizations and conclusions through *inductive inference,* which thereby introduces an element of uncertainty. An inductive inference based on experimental evidence can never be absolutely certain; at best, it may be highly probable. This thought was expressed by Einstein when he said, in effect, that a hundred successful experiments can never prove that a theory is correct, but a single contradiction will prove that it is incorrect. On the other hand, in geometry (as in all pure mathematics) we establish generalizations through *deductive inferences.* This is a procedure which demands that *some* propositions or generalizations are mutually agreed upon *at the very outset,* as are also conventional rules of thought known as formal logic. In this way, *no uncertainty enters* the picture. If you accept the assumed propositions and the conventional mode of logical reasoning, then you must also accept the conclusions which are thus arrived at by deduction, or implication.

Such is the central theme which is very ably expounded in the present article.

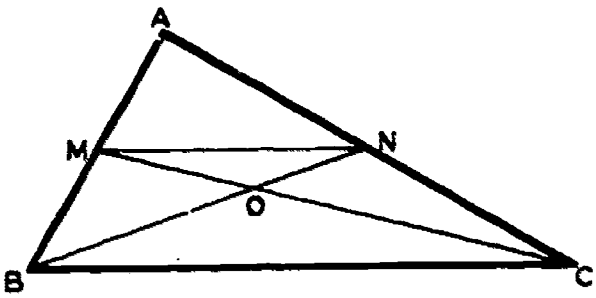
GEOMETRY AND EMPIRICAL SCIENCE

C. G. HEMPEL

1. INTRODUCTION. The most distinctive characteristic which differentiates mathematics from the various branches of empirical science, and which accounts for its fame as the queen of the sciences, is no doubt the peculiar certainty and necessity of its results. No proposition in even the most advanced parts of empirical science can ever attain this status; a hypothesis concerning "matters of empirical fact" can at best acquire what is loosely called a high probability or a high degree of confirmation on the basis of the relevant evidence available; but however well it may have been confirmed by careful tests, the possibility can never be precluded that it will have to be discarded later in the light of new and disconfirming evidence. Thus, all the theories and hypotheses of empirical science share this provisional character of being established and accepted "until further notice," whereas a mathematical theorem, once proved, is established once and for all; it holds with that particular certainty which no subsequent empirical discoveries, however unexpected and extraordinary, can ever affect to the slightest extent. It is the purpose of this paper to examine the nature of that proverbial "mathematical certainty" with special reference to geometry, in an attempt to shed some light on the question as to the validity of geometrical theories, and their significance for our knowledge of the structure of physical space.

The nature of mathematical truth can be understood through an analysis of the method by means of which it is established. On this point I can be very brief: it is the method of mathematical demonstration, which consists in the logical deduction of the proposition to be proved from other propositions, previously established. Clearly, this procedure would involve an infinite regress unless some propositions were accepted without proof; such propositions are indeed found in every mathematical discipline which is rigorously developed; they are the *axioms* or *postulates* (we shall use these terms interchangeably) of the theory. Geometry provides the historically first example of the axiomatic presentation of a mathematical discipline. The classical set of postulates, however, on which Euclid based his system, has proved insufficient for the deduction of the well-known theorems of so-called euclidean geometry; it has therefore been revised and supplemented in modern times, and at present various adequate systems of postulates for euclidean geometry are available; the one most closely related to Euclid's system is probably that of Hilbert.

2. THE INADEQUACY OF EUCLID'S POSTULATES. The inadequacy of Euclid's own set of postulates illustrates a point which is crucial for the axiomatic method in modern mathematics: Once the postulates for a theory have been laid down, every further proposition of the theory must be proved exclusively by logical deduction from the postulates; any appeal, explicit or implicit, to a feeling of self-evidence, or to the characteristics of geometrical figures, or to our experiences concerning the behavior of rigid bodies in physical space, or the like, is strictly prohibited; such devices may have a heuristic value in guiding our efforts to find a strict proof for a theorem, but the proof itself must contain absolutely no reference to such aids. This is particularly important in geometry, where our so-called intuition of geometrical relationships, supported by reference to figures or to previous physical experiences, may induce us tacitly to make use of assumptions which are neither formulated in our postulates nor provable by means of them. Consider, for example, the theorem that in a triangle the three medians bisecting the sides intersect in one point which divides each of them in the ratio of 1:2 To prove this theorem, one shows first that in any triangle ABC (see figure) the line segment MN which connects the centers of AB and AC is parallel to BC and therefore half as long as the latter side. Then the lines BN and CM are drawn, and an examination of the triangles MON and BOC leads to the proof of the theorem. In this procedure, it is usually taken for granted that BN and CM intersect in a point O which lies



between B and N as well as between C and M. This assumption is based on geometrical intuition, and indeed, it cannot be deduced from Euclid's postulates: to make it strictly demonstrable and independent of any reference to intuition, a special group of postulates has been added to those of Euclid; they are the postulates of order. One of these—to give an example—asserts that if A, B, C are points on a straight line l, and if B lies between A and C, then B also lies between C and A.—Not even as "trivial" an assumption as this may be taken for granted; the system of postulates has to be made so complete that all the required propositions can be deduced from it by purely logical means.

Another illustration of the point under consideration is provided by the proposition that triangles which agree in two sides and the enclosed angle, are congruent. In Euclid's Elements, this proposition is presented as a theorem; the alleged proof, however, makes use of the ideas of motion and superimposition of figures and thus involves tacit assumptions which are based on our geometric intuition and on experiences with rigid bodies, but which are definitely not warranted by—*i.e.* deducible from—Euclid's postulates. In Hilbert's system, therefore, this proposition (more precisely: part of it) is explicitly included among the postulates.

3. MATHEMATICAL CERTAINTY. It is this purely deductive character of mathematical proof which forms the basis of mathematical certainty: What the rigorous proof of a theorem—say the proposition about the sum of the angles in a triangle—establishes is not the truth of the proposition in question but rather a conditional insight to the effect that that proposition is certainly true *provided that* the postulates are true; in other words, the proof of a mathematical proposition establishes the fact that the latter is logically implied by the postulates of the theory in question. Thus, each mathematical theorem can be cast into the form.

$$(P_1 \cdot P_2 \cdot P_3 \cdot \cdots \cdot P_N) \rightarrow T$$

where the expression on the left is the conjunction (joint assertion) of all the postulates, the symbol on the right represents the theorem in its customary formulation, and the arrow expresses the relation of logical implication or entailment. Precisely this character of mathematical theorems is the reason for their peculiar certainty and necessity, as I shall now attempt to show.

It is typical of any purely logical deduction that the conclusion to which it leads simply re-asserts (a proper or improper) part of what has already been stated in the premises. Thus, to illustrate this point by a very elementary example, from the premise, "This figure is a right triangle," we can deduce the conclusion, "This figure is a triangle"; but this conclusion clearly reiterates part of the information already contained in the premise. Again, from the premises, "All primes different from 2 are odd" and "n is a prime different from 2," we can infer logically that n is odd; but this consequence merely repeats part (indeed a relatively small part) of the information contained in the premises. The same situation prevails in all other cases of logical deduction; and we may, therefore, say that logical deduction—which is the one and only method of mathematical proof—is a technique of conceptual analysis: it discloses what assertions are concealed in a given set of premises, and it

makes us realize to what we committed ourselves in accepting those premises; but none of the results obtained by this technique ever goes by one iota beyond the information already contained in the initial assumptions.

Since all mathematical proofs rest exclusively on logical deductions from certain postulates, it follows that a mathematical theorem, such as the Pythagorean theorem in geometry, asserts nothing that is *objectively* or *theoretically new* as compared with the postulates from which it is derived, although its content may well be *psychologically new* in the sense that we were not aware of its being implicitly contained in the postulates.

The nature of the peculiar certainty of mathematics is now clear: A mathematical theorem is certain *relatively* to the set of postulates from which it is derived; *i.e.* it is necessarily true *if* those postulates are true; and this is so because the theorem, if rigorously proved, simply re-asserts part of what has been stipulated in the postulates. A truth of this conditional type obviously implies no assertions about matters of empirical fact and can, therefore, never get into conflict with any empirical findings, even of the most unexpected kind; consequently, unlike the hypotheses and theories of empirical science, it can never suffer the fate of being disconfirmed by new evidence: A mathematical truth is irrefutably certain just because it is devoid of factual, or empirical content. Any theorem of geometry, therefore, when cast into the conditional form described earlier, is analytic in the technical sense of logic, and thus true *a priori*: *i.e.* its truth can be established by means of the formal machinery of logic alone, without any reference to empirical data.

4. POSTULATES AND TRUTH. Now it might be felt that out analysis of geometrical truth so far tells only half of the relevant story. For while a geometrical proof no doubt enables us to assert a proposition conditionally—namely on condition that the postulates are accepted—, is it not correct to add that geometry also unconditionally asserts the truth of its postulates and thus, by virtue of the deductive relationship between postulates and theorems, enables us unconditionally to assert the truth of its theorems? Is it not an unconditional assertion of geometry that two points determine one and only one straight line that connects them, or that in any triangle, the sum of the angles equals two right angles? That this is definitely not the case, is evidenced by two important aspects of the axiomatic treatment of geometry which will now be briefly considered.

The first of these features is the well-known fact that in the more recent development of mathematics, several systems of geometry have been constructed which are incompatible with euclidean geometry, and

in which, for example, the two propositions just mentioned do not necessarily hold. Let us briefly recollect some of the basic facts concerning these *non-euclidean geometries*. The postulates on which euclidean geometry rests include the famous postulate of the parallels, which, in the case of plane geometry, asserts in effect that through every point P not on a given line l there exists exactly one parallel to l, *i.e.*, one straight line which does not meet l. As this postulate is considerably less simple than the others, and as it was also felt to be intuitively less plausible than the latter, many efforts were made in the history of geometry to prove that this proposition need not be accepted as an axiom, but that it can be deduced as a theorem from the remaining body of postulates. All attempts in this direction failed, however; and finally it was conclusively demonstrated that a proof of the parallel principle on the basis of the other postulates of euclidean geometry (even in its modern, completed form) is impossible. This was shown by proving that a perfectly self-consistent geometrical theory is obtained if the postulate of the parallels is replaced by the assumption that through any point P not on a given straight line l there exist at least two parallels to l. This postulate obviously contradicts the euclidean postulate of the parallels, and if the latter were actually a consequence of the other postulates of euclidean geometry, then the new set of postulates would clearly involve a contradiction, which can be shown not to be the case. This first non-euclidean type of geometry, which is called hyperbolic geometry, was discovered in the early 20's of the last century almost simultaneously, but independently by the Russian N. I. Lobatschefskij, and by the Hungarian J. Bolyai. Later, Riemann developed an alternative geometry, known as elliptical geometry, in which the axiom of the parallels is replaced by the postulate that no line has any parallels. (The acceptance of this postulate, however, in contradistinction to that of hyperbolic geometry, requires the modification of some further axioms of euclidean geometry, if a consistent new theory is to result.) As is to be expected, many of the theorems of these non-euclidean geometries are at variance with those of euclidean theory; thus, *e.g.*, in the hyperbolic geometry of two dimensions, there exist, for each straight line l, through any point P not on l, infinitely many straight lines which do not meet l; also, the sum of the angles in any triangle is less than two right angles. In elliptic geometry, this angle sum is always greater than two right angles; no two straight lines are parallel; and while two different points usually determine exactly one straight line connecting them (as they always do in euclidean geometry), there are certain pairs of points which are connected by infinitely many different straight lines. An illustration of this latter type of geometry is provided by the geometrical structure of that curved two-dimensional space which is represented by the surface of a sphere, when the concept of straight

line is interpreted by that of great circle on the sphere. In this space, there are no parallel lines since any two great circles intersect; the endpoints of any diameter of the sphere are points connected by infinitely many different "straight lines," and the sum of the angles in a triangle is always in excess of two right angles. Also, in this space, the ratio between the circumference and the diameter of a circle (not necessarily a great circle) is always less than 2π.

Elliptic and hyperbolic geometry are not the only types of non-euclidean geometry; various other types have been developed; we shall later have occasion to refer to a much more general form of non-euclidean geometry which was likewise devised by Riemann.

The fact that these different types of geometry have been developed in modern mathematics shows clearly that mathematics cannot be said to assert the truth of any particular set of geometrical postulates; all that pure mathematics is interested in, and all that it can establish, is the deductive consequences of given sets of postulates and thus the necessary truth of the ensuing theorems relatively to the postulates under consideration.

A second observation which likewise shows that mathematics does not assert the truth of any particular set of postulates refers to *the status of the concepts in geometry*. There exists, in every axiomatized theory, a close parallelism between the treatment of the propositions and that of the concepts of the system. As we have seen, the propositions fall into two classes: the postulates, for which no proof is given, and the theorems, each of which has to be derived from the postulates. Analogously, the concepts fall into two classes: the primitive or basic concepts, for which no definition is given, and the others, each of which has to be precisely defined in terms of the primitives. (The admission of some undefined concepts is clearly necessary if an infinite regress in definition is to be avoided.) The analogy goes farther: Just as there exists an infinity of theoretically suitable axiom systems for one and the same theory—say, euclidean geometry—, so there also exists an infinity of theoretically possible choices for the primitive terms of that theory; very often—but not always—different axiomatizations of the same theory involve not only different postulates, but also different sets of primitives. Hilbert's axiomatization of plane geometry contains six primitives: point, straight line, incidence (of a point on a line), betweenness (as a relation of three points on a straight line), congruence for line segments, and congruence for angles. (Solid geometry, in Hilbert's axiomatization, requires two further primitives, that of plane and that of incidence of a point on a plane.) All other concepts of geometry, such as those of angle, triangle, circle, *etc.*, are defined in terms of these basic concepts.

But if the primitives are not defined within geometrical theory, what meaning are we to assign to them? The answer is that it is entirely unnecessary to connect any particular meaning with them. True, the words "point," "straight line," *etc.*, carry definite connotations with them which relate to the familiar geometrical figures, but the validity of the propositions is completely independent of these connotations. Indeed, suppose that in axiomatized euclidean geometry, we replace the oversuggestive terms "point," "straight line," "incidence," "betweenness," *etc.*, by the neutral terms "object of kind 1," "object of kind 2," "relation No. 1," "relation No. 2," *etc.*, and suppose that we present this modified wording of geometry to a competent mathematician or logician who, however, knows nothing of the customary connotations of the primitive terms. For this logician, all proofs would clearly remain valid, for as we saw before, a rigorous proof in geometry rests on deduction from the axioms alone without any reference to the customary interpretation of the various geometrical concepts used. We see therefore that indeed no specific meaning has to be attached to the primitive terms of an axiomatized theory; and in a precise logical presentation of axiomatized geometry the primitive concepts are accordingly treated as so-called logical variables.

As a consequence, geometry cannot be said to assert the truth of its postulates, since the latter are formulated in terms of concepts without any specific meaning; indeed, for this very reason, the postulates themselves do not make any specific assertion which could possibly be called true or false! In the terminology of modern logic, the postulates are not sentences, but sentential functions with the primitive concepts as variable arguments.—This point also shows that the postulates of geometry cannot be considered as "self-evident truths," because where no assertion is made, no self-evidence can be claimed.

5. Pure and physical geometry. Geometry thus construed is a purely formal discipline; we shall refer to it also as *pure geometry*. A pure geometry, then,—no matter whether it is of the euclidean or of a non-euclidean variety—deals with no specific subject-matter; in particular, it asserts nothing about physical space. All its theorems are analytic and thus true with certainty precisely because they are devoid of factual content. Thus, to characterize the import of pure geometry, we might use the standard form of a movie-disclaimer: No portrayal of the characteristics of geometrical figures or of the spatial properties or relationships of actual physical bodies is intended, and any similarities between the primitive concepts and their customary geometrical connotations are purely coincidental.

But just as in the case of some motion pictures, so in the case at least of euclidean geometry, the disclaimer does not sound quite convincing:

Historically speaking, at least, euclidean geometry has its origin in the generalization and systematization of certain empirical discoveries which were made in connection with the measurement of areas and volumes, the practice of surveying, and the development of astronomy. Thus understood, geometry has factual import; it is an empirical science which might be called, in very general terms, the theory of the structure of physical space, or briefly, *physical geometry*. What is the relation between pure and physical geometry?

When the physicist uses the concepts of point, straight line, incidence, *etc.*, in statements about physical objects, he obviously connects with each of them a more or less definite physical meaning. Thus, the term "point" serves to designate physical points, *i.e.*, objects of the kind illustrated by pin-points, cross hairs, *etc.* Similarly, the term "straight line" refers to straight lines in the sense of physics, such as illustrated by taut strings or by the path of light rays in a homogeneous medium. Analogously, each of the other geometrical concepts has a concrete physical meaning in the statements of physical geometry. In view of this situation, we can say that physical geometry is obtained by what is called, in contemporary logic, a semantical interpretation of pure geometry. Generally speaking, a semantical interpretation of a pure mathematical theory, whose primitives are not assigned any specific meaning, consists in giving each primitive (and thus, indirectly, each defined term) a specific meaning or designatum. In the case of physical geometry, this meaning is physical in the sense just illustrated; it is possible, however, to assign a purely arithmetical meaning to each concept of geometry; the possibility of such an arithmetical interpretation of geometry is of great importance in the study of the consistency and other logical characteristics of geometry, but it falls outside the scope of the present discussion.

By virtue of the physical interpretation of the originally uninterpreted primitives of a geometrical theory, physical meaning is indirectly assigned also to every defined concept of the theory; and if every geometrical term is now taken in its physical interpretation, then every postulate and every theorem of the theory under consideration turns into a statement of physics, with respect to which the question as to truth or falsity may meaningfully be raised—a circumstance which clearly contradistinguishes the propositions of physical geometry from those of the corresponding uninterpreted pure theory.—Consider, for example, the following postulate of pure euclidean geometry: For any two objects x, y of kind 1, there exists exactly one object l of kind 2 such that both x and y stand in relation No. 1 to l. As long as the three primitives occurring in this postulate are uninterpreted, it is obviously meaningless to ask whether the postulate is true. But by virtue of the above physical inter-

pretation, the postulate turns into the following statement: For any two physical points x, y there exists exactly one physical straight line l such that both x and y lie on l. But this is a physical hypothesis, and we may now meaningfully ask whether it is true or false. Similarly, the theorem about the sum of the angles in a triangle turns into the assertion that the sum of the angles (in the physical sense) of a figure bounded by the paths of three light rays equals two right angles.

Thus, the physical interpretation transforms a given pure geometrical theory—euclidean or non-euclidean—into a system of physical hypotheses which, if true, might be said to constitute a theory of the structure of physical space. But the question whether a given geometrical theory in physical interpretation is factually correct represents a problem not of pure mathematics but of empirical science: it has to be settled on the basis of suitable experiments or systematic observations. The only assertion the mathematician can make in this context is this: If all the postulates of a given geometry, in their physical interpretation, are true, then all the theorems of that geometry, in their physical interpretation, are necessarily true, too, since they are logically deducible from the postulates. It might seem, therefore, that in order to decide whether physical space is euclidean or non-euclidean in structure, all that we have to do is to test the respective postulates in their physical interpretation. However, this is not directly feasible; here, as in the case of any other physical theory, the basic hypotheses are largely incapable of a direct experimental test; in geometry, this is particularly obvious for such postulates as the parallel axiom or Cantor's axiom of continuity in Hilbert's system of euclidean geometry, which makes an assertion about certain infinite sets of points on a straight line. Thus, the empirical test of a physical geometry no less than that of any other scientific theory has to proceed indirectly; namely, by deducing from the basic hypotheses of the theory certain consequences, or predictions, which are amenable to an experimental test. If a test bears out a prediction, then it constitutes confirming evidence (though, of course, no conclusive proof) for the theory; otherwise, it disconfirms the theory. If an adequate amount of confirming evidence for a theory has been established, and if no disconfirming evidence has been found, then the theory may be accepted by the scientist "until further notice."

It is in the context of this indirect procedure that pure mathematics and logic acquire their inestimable importance for empirical science: While formal logic and pure mathematics do not in themselves establi any assertions about matters of empirical fact, they provide an efficient and entirely indispensable machinery for deducing, from abstract theoretical assumptions, such as the laws of Newtonian mechanics or the pos-

tulates of euclidean geometry in physical interpretation, consequences concrete and specific enough to be accessible to direct experimental test. Thus, *e.g.*, pure euclidean geometry shows that from its postulates there may be deduced the theorem about the sum of the angles in a triangle, and that this deduction is possible no matter how the basic concepts of geometry are interpreted; hence also in the case of the physical interpretation of euclidean geometry. This theorem, in its physical interpretation, is accessible to experimental test; and since the postulates of elliptic and of hyperbolic geometry imply values different from two right angles for the angle sum of a triangle, this particular proposition seems to afford a good opportunity for a crucial experiment. And no less a mathematician than Gauss did indeed perform this test; by means of optical methods – and thus using the interpretation of physical straight lines as paths of light rays – he ascertained the angle sum of a large triangle determined by three mountain tops. Within the limits of experimental error, he found it equal to two right angles.

6. ON POINCARÉ'S CONVENTIONALISM CONCERNING GEOMETRY. But suppose that Gauss had found a noticeable deviation from this value; would that have meant a refutation of euclidean geometry in its physical interpretation, or, in other words, of the hypothesis that physical space is euclidean in structure? Not necessarily; for the deviation might have been accounted for by a hypothesis to th- effects that the paths of the light rays involved in the sighting process were bent by some disturbing force and thus were not actually straight lines. The same kind of reference to deforming forces could also be us- l if, say, the eculidean theorems of congruence for plane figures were tested in their physical interpretation by means of experiments involving rigid bodies, and if any violations of the theorems were found. This point is by no means trivial; Henri Poincaré, the great French mathematician and theoretical physicist, based on considerations of this type his famous *conventionalism concerning geometry*. It was his opinion that no empirical test, whatever its outcome, can conclusively invalidate the euclidean conception of physical space; in other words, the validity of euclidean geometry in physical science can always be preserved – if necessary, by suitable changes in the theories of physics, such as the introduction of new hypotheses concerning deforming or deflecting forces. Thus, the question as to whether physical space has a euclidean or a non-euclidean structure would become a matter of convention, and the decision to preserve euclidean geometry at all costs would recommend itself, according to Poincaré, by the greater simplicity of euclidean as compared with non-euclidean geometrical theory.

It appears, however, that Poincaré's account is an oversimplification.

It rightly calls attention to the fact that the test of a physical geometry G always presupposes a certain body P of non-geometrical physical hypotheses (including the physical theory of the instruments of measurement and observation used in the test), and that the so-called test of G actually bears on the combined theoretical system $G{\cdot}P$ rather than on G alone. Now, if predictions derived from $G{\cdot}P$ are contradicted by experimental findings, then a change in the theoretical structure becomes necessary. In classical physics, G always was euclidean geometry in its physical interpretation, GE; and when experimental evidence required a modification of the theory, it was P rather than GE which was changed. But Poincaré's assertion that this procedure would always be distinguished by its greater simplicity is not entirely correct; for what has to be taken into consideration is the simplicity of the total system $G{\cdot}P$, and not just that of its geometrical part. And here it is clearly conceivable that a simpler total theory in accordance with all the relevant empirical evidence is obtainable by going over to a non-euclidean form of geometry rather than by preserving the euclidean structure of physical space and making adjustments only in part P.

And indeed, just this situation has arisen in physics in connection with the development of the general theory of relativity: If the primitive terms of geometry are given physical interpretations along the lines indicated before, then certain findings in astronomy represent good evidence in favor of a total physical theory with a non-euclidean geometry as part G. According to this theory, the physical universe at large is a three-dimensional curved space of a very complex geometrical structure; it is finite in volume and yet unbounded in all directions. However, in comparatively small areas, such as those involved in Gauss' experiment, euclidean geometry can serve as a good approximative account of the geometrical structure of space. The kind of structure ascribed to physical space in this theory may be illustrated by an analogue in two dimensions; namely, the surface of a sphere. The geometrical structure of the latter, as was pointed out before, can be described by means of elliptic geometry, if the primitive term "straight line" is interpreted as meaning "great circle," and if the other primitives are given analogous interpretations. In this sense, the surface of a sphere is a two-dimensional curved space of non-euclidean structure, whereas the plane is a two-dimensional space of euclidean structure. While the plane is unbounded in all directions, and infinite in size, the spherical surface is finite in size and yet unbounded in all directions: a two-dimensional physicist, travelling along "straight lines" of that space would never encounter any boundaries of his space; instead, he would finally return to his point of departure, provided that his life span and his technical facilities were suffi-

cient for such a trip in consideration of the size of his "universe." It is interesting to note that the physicists of that world, even if they lacked any intuition of a three-dimensional space, could empirically ascertain the fact that their two-dimensional space was curved. This might be done by means of the method of travelling along straight lines: another, simpler test would consist in determining the angle sum in a triangle; again another in determining, by means of measuring tapes, the ratio of the circumference of a circle (not necessarily a great circle) to its diameter; this ratio would turn out to be less than π.

The geometrical structure which relativity physics ascribes to physical space is a three-dimensional analogue to that of the surface of a sphere, or, to be more exact, to that of the closed and finite surface of a potato, whose curvature varies from point to point. In our physical universe, the curvature of space at a given point is determined by the distribution of masses in its neighborhood; near large masses such as the sun, space is strongly curved, while in regions of low mass-density, the structure of the universe is approximately euclidean. The hypothesis stating the connection between the mass distribution and the curvature of space at a point has been approximately confirmed by astronomical observations concerning the paths of light rays in the gravitational field of the sun.

The geometrical theory which is used to describe the structure of the physical universe is of a type that may be characterized as a generalization of elliptic geometry. It was originally constructed by Riemann as a purely mathematical theory, without any concrete possibility of practical application at hand. When Einstein, in developing his general theory of relativity, looked for an appropriate mathematical theory to deal with the structure of physical space, he found in Riemann's abstract system the conceptual tool he needed. This fact throws an interesting sidelight on the importance for scientific progress of that type of investigation which the "practical-minded" man in the street tends to dismiss as useless, abstract mathematical speculation.

Of course, a geometrical theory in physical interpretation can never be validated with mathematical certainty, no matter how extensive the experimental tests to which it is subjected; like any other theory of empirical science, it can acquire only a more or less high degree of confirmation. Indeed, the considerations presented in this article show that the demand for mathematical certainty in empirical matters is misguided and unreasonable; for, as we saw, mathematical certainty of knowledge can be attained only at the price of analyticity and thus of complete lack of factual content. Let me summarize this insight in Einstein's words:

"As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."

FOREWORD

The present essay by Professor Lenzen, although written a little over a quarter of a century ago, is as valid today as when first published. Moreover, the article affords an excellent historical perspective to the development of geometry and its relation to physical science and to the concept of measurement.

It has been said that much nonsense has been written about the nature of measurement. To be sure, the ancient Greek geometers, and indeed, Western mathematicians until the middle of the 17th century, talked and thought in terms of *lengths* and *areas* as if these were basic realities. But when Descartes expressed the *distance* between two points analytically, that is, in terms of coordinates, he transformed a physical reality into an idealized model; an expression involving *numbers* enabled him to dispense with the geometrical figure which represented the "length" of a segment.

The full significance of this breakthrough is vividly exemplified in contemporary physics. It is clear today that the essence of measurement in general consists of mapping empirical observations and relations into an appropriate formal mathematical model. Ironically, many centuries of mathematical development were required to arrive at this concept. It is a point of view which had to wait until the 20th century—until mathematics had become universally regarded as a disclipline which includes many postulational systems.

PHYSICAL GEOMETRY

V. F. LENZEN

1. INTRODUCTION. Prior to Einstein a distinction was usually made between geometry and physics. Geometry was viewed as a rational science which is independent of sensory experience; physics was known to be an empirical science based upon observation and experiment. The sharp separation between mathematics and physics may be illustrated by the sciences of kinematics and dynamics. In his *Principles of Mechanics*, which was published in 1905, Slate says, "In the first two chapters we shall be occupied with conceptions—Velocity and Acceleration—that rest entirely upon a mathematical basis. . . . If mechanics is taken to include kinematics also, as it frequently is, that part of the science which is physical and not geometrical must be specially distinguished. It is designated as Dynamics. The point should be watched at which the transition is . . . made by introducing experimental results into the framework of our science." The ideas expressed by Slate are characteristic of older books on mechanics. In the study of motion there was recognized the progression: geometry, the science of space; kinematics, the science of motion which was based upon the addition of time to space; dynamics or mechanics, which explained the motions of the material bodies in the physical world. Geometry and kinematics were viewed as mathematical sciences, dynamics or mechanics as a physical science. In the present paper I shall show how geometry and physics have been united in the science of physical geometry.

2. HISTORICAL SKETCH. Our discussion of the relation of geometry to physics may well be prefaced by a description of its subject matter. Geometry is frequently defined as the science of space, but what is space? One of the best answers to this question is given in Carnap's early monograph, *Der Raum* [1]. In this work he distinguishes between formal space, intuitional space, and physical space. Formal space is a system of general ordinal relations. The formal properties of the terms and relations of such a structure are determined by postulates. Formal or abstract space is the subject matter of abstract geometry. Intuitional space is the system of relations between spatial objects such as lines, surfaces, and volumes, the properties of which are apprehended in sense-

perception or imagination. Intuitional space is especially considered in the Kantian philosophy of geometry. Physical space is the system of relations between the bodies and phenomena of the physical world and is the subject matter of physical geometry. It may be added that the distinction between topological, projective, and metrical properties applies equally to formal space, intuitional space, and physical space. The present discussion will find need only for abstract geometry and physical geometry.

The development of an understanding of the relation between geometry and physics may be credited principally to the theory of relativity. This theory initiated a program for the reduction of physics to geometry. The special theory of relativity made it possible to express kinematics in terms of a four-dimensional space-time. In the general theory, space-time is viewed as a Riemannian continuum whose curvature is determined by matter. A free material particle describes a world line which is a geodesic of this continuum. The general theory of relativity thus reduces the physics of gravitation to geometry, and unified field theories have been constructed in order to reduce all physics to geometry. This geometrization of physics appears to have made it a branch of mathematics, to have freed it from dependence on experience. A unified mathematical representation of physical phenomena is offered, and this achievement has inspired Sir James Jeans to declare that God is to be conceived as a pure mathematician.

The reduction of physics to geometry requires, however, that geometry be exhibited as an émpirical science. In so far as geometry can be applied to the physical world it is based upon observation and experiment. I shall represent geometry to be the most firmly established branch of physics. If physics is to be reduced to geometry, geometry must also be reduced to physics.

That the concept of physical geometry is a significant contribution may be shown by exhibiting historical philosophical interpretations of geometry. Geometry as a mathematical science was created by the ancient Greeks, but the raw materials for a geometry were fashioned by their predecessors, notably the Egyptians. The Egyptians had to make surveys of land in order to redetermine the marks of boundaries which had been washed away by the floods of the Nile. Hence they measured distances and lengths and discovered propositions that express the metrical relations of the elements of simple figures. The Egyptians thus discovered and used propositions of physical geometry. The Greeks organized such propositions into a deductive science; Euclid founded geometry upon axioms and postulates from which propositions may be

derived as theorems. Euclidean geometry has furnished the classical model for science.

The Greeks created the deductive science of geometry and originated the view that geometry is a rational science which is independent of sensory experience. Thus Plato taught that the objects of science must be universal and permanent. The objects of perception are in a state of flux, and hence propositions about the world of experience are infected with uncertainty and relativity. He explained the possibility of rational science by the theory of a transcendent world of pure forms, or ideas, which can be known only by reason. Geometrical structures such as triangles and circles are pure forms which are to be distinguished from the crude perceptible triangles and circles in the world of sense-perception. Geometry is approximately applicable to experience because perceptible figures participate in the pure forms. The soul has direct knowledge of pure forms in a pre-earthly state of existence; perception through the senses stimulates recollection of the pure forms in which the objects of perception participate. In support of his theory that knowledge of geometrical figures is latent in the individual mind, Plato narrates how Socrates guides an uneducated slave boy step by step to the recognition of the truth of a proposition in geometry. Thus the Platonic philosophy of geometry interpreted the objects of geometry to be ideal entities which transcend ordinary experience.

Since the eighteenth century the theory of Kant has exerted a widespread influence. Kant started from the assumption that pure mathematics, which is exemplified by geometry, is *a priori* and therefore independent of experience. He propounded the question, how is pure mathematics possible? His answer as applied to geometry was that space is the *a priori* form of external intuition which is the condition of all perceptual experience. Geometrical figures are constructions in space and can be constructed in pure intuition independently of sensory experience. This theory provided a new foundation for the interpretation of geometry as the science of universal and necessary truths.

The Kantian theory dominated the philosophy of geometry during the nineteenth century. Geometrical figures were assumed to be constructed in pure intuition and analysis of such figures yielded the self-evident axioms of Euclidean geometry. In recent years the German philosopher Husserl has offered intuitions into the essence of geometrical structures as the foundation of geometry. Intuitional space which is referred to by Carnap, is an inheritance from Kant. During the nineteenth century, however, the non-Euclidean geometries were created and led to the development of new points of view. Helmholtz and others exhibited intuitive models of the non-Euclidean geometries, and thus

shook the Kantian doctrine that intuition reveals physical space to be Euclidean. The study of foundations led to the abstract theory of geometry, according to which the propositions of geometry are blank forms devoid of empirical reference. The postulates of a geometry constitute an implicit definition of the fundamental concepts which express the properties of formal space. Geometrical theory is concerned with the deductive dependence of theorems upon postulates. Since postulates and theorems are devoid of empirical significance, the problem of their truth or falsity does not arise. A proposition in geometry becomes true or false only when a concrete interpretation is given to the concepts.

The criticism of the theory that pure intuition is the origin of geometry was accompanied by the development of the view that in so far as geometry can be used in physics, geometrical propositions express the positional relations of perceptible bodies. Gauss measured the angles of a physical triangle whose sides were light rays, in order to test whether or not the sum of the angles is equal to two right angles. Helmholtz [2] in an essay on the origin and significance of the axioms of geometry declared that these axioms describe the mechanical behavior of our most rigid bodies during motions. Riemann [3] in his famous essay on the hypotheses which constitute the foundations of geometry advanced the hypothesis that the metrical structure of physical space depends on the physical forces in it. Thus the question, is physical space Euclidean or non-Euclidean?, acquired significance. The significance of this question presupposes that the metrical structure of space is defined in terms of the positional relations of physical bodies or phenomena. The standpoints of Gauss, Helmholtz and Riemann eventually were realized in the contemporary concept of physical geometry which is exemplified in Einstein's relativistic theory of gravitation. Geometry, in so far as it is relevant to physics, is a physical science that is based upon observation and experiment.

3. An operational theory. Synthetic treatment. The function of physical geometry is to describe the properties of physical space. In preparation for an exposition of how physical geometry may be developed, it is desirable to set forth the elements of the problem. In agreement with Carnap. I distinguish data of experience, postulate of measure, and relational structure. Data of experience are the contact of two points at a specific time, the incidence of a point on a line, the inclusion of a body by a surface, and so forth. Perceptions of contact, or of coincidence, especially furnish the raw materials of geometry. But such data of experience are sufficient only for the topological structure of space. Projective properties require the determination of straight lines, and metrical properties require procedures for measuring length and angles.

Projective and metrical geometry are relative to definitions which are matters of convention. Carnap has clearly shown that it is possible to proceed in two ways. One may adopt a postulate of measure and then by observation determine the scheme of geometrical relations that describes the metrical structure of space. Experience determines whether physical space is Euclidean or non-Euclidean only if a standard of measure has been adopted. It has been traditional to adopt as standard of measure the distance between two points on a rigid body and to postulate that this distance is independent of position. As Carnap has pointed out, an alternative procedure is to postulate the scheme of geometrical relations and then determine from experience the standard of measure that is implied. The possibility of this procedure was especially emphasized by Poincaré, who declared that geometry is determined by conventional definitions. He contended that since Euclidean geometry is the simplest, convention will decree its continued employment for the description of physical phenomena. If light did not travel in straight lines, Euclidean geometry could still be used to formulate different laws of physics. The general theory of relativity, however, predicts a behavior of rigid bodies which makes it convenient to change the geometry rather than the standard of measure.

The foregoing discussion demonstrates that metrical physical geometry exemplifies the operational theory of physical concepts. This theory, which has been expounded notably by Bridgman [4], expresses the meaning of physical concepts in terms of operations. In order to measure a physical quantity it is necessary to control the conditions under which a quantity assumes a determinate value. The procedures of measurement require physical and mental operations that are performed in accordance with prescribed rules. The definition of a physical quantity is expressed by the description of the conditions and procedures of measurement. Consistent application of this operational theory leads to the interpretation of a physical quantity as a number assigned to a physical property of bodies. Thus the definition of a physical quantity does not express an intuitive insight into an intrinsic essence of the quantity. Textbooks of physics have defined mass as the quantity of matter in a body, but this is only a verbal definition. A significant definition of mass must describe the procedure for measuring the mass of a body. The same point of view applies to physical geometry. Consider, for example, the concept of length. Some philosophers have declared that we have a direct perception of length which acquaints us with the meaning of the concept; this has been the basis for the concept of intuitional space. The operational theory, however, recognizes that length as a physical quantity depends on operations of measurement in terms of a standard. The

operational nature of length is especially demonstrated in the special theory of relativity.

I have several times referred to a standard of measure as a basis for metrical physical geometry. This standard is based upon the properties of practically rigid bodies. I assume that we are acquainted with examples of such bodies: sticks, stones, and manufactured bodies, such as iron rods. In order to describe the properties of rigid bodies, let us suppose that two points have been made on such a body. A point will be a hole made by a pin or a dot with a pencil. The two points may be called a rigid point-pair and determine a stretch. Given two rigid point-pairs that may be placed alongside each other so that the points of one are in contact with the points of the other. If the rigid pairs are displaced together, the contacts are preserved. If one rigid pair is kept fixed and the other displaced and returned to its initial position, the contacts are restored. If a number of rigid point-pairs can be brought consecutively into contact with a specific pair, they can be brought into contact with one another. Stretches defined by rigid point-pairs in contact are said to be congruent. If it is postulated that the length of a stretch is independent of position, stretches at a distance may be defined to be congruent.

I shall now explain how metrical, physical geometry may be developed so as to describe the properties of physical space. Physical space may be defined as the system of positional relations of perceptible bodies and phenomena. Such positional relations may be investigated from the standpoint of topology, but I propose to study the metrical structure of space. For this purpose we adopt rigid point-pairs as standards of measure. Thus the metrical structure is determined by the positional relations of practically rigid bodies. Indeed, Einstein [5] has described space as the totality of possibilities of relative position of practically rigid bodies.

On investigating the properties of space it is necessary to specify a frame of reference relative to which rigid bodies are at rest or in motion. In elementary geometry a geometrical structure is ordinarily assumed to be at rest in a frame that is rigidly attached to the earth. As we shall see, however, the special theory of relativity has brought to light the relativity of space to a frame of reference.

The procedure in building physical geometry is exemplified by some elementary experiments which have been described by Carnap [1, p. 41]. Let us have given a standard body of which two points A, B determine a standard stretch. Consider a physical surface such as the top of a desk.

(1) We discover that A and B and also C and D of the standard body

may be brought simultaneously into contact with four points A_1, B_1, C_1, D_1, upon the surface. Repeated experiments demonstrate that whenever A, B, C or A, C, D or B, C, D are in contact with their corresponding points, the fourth pair of points is in contact. The pair A, B can be brought into contact with B_1 and C_1, with C_1 and D_1 and with D_1 and B_1. The conclusion is that with respect to the point-pair (A, B) as a standard, A_1, B_1; B_1, C_1; C_1, D_1; D_1, B_1 are rigid point-pairs. From the first experiment one infers the rigidity of C, D and further the rigidity of the set A, B, C, D.

(2) If A, B, C, D are brought into contact with four other points of the surface, repeated experiments yield the same results as before, and therefore the other points constitute a rigid set of points. All sets of four points of the surface are demonstrated to be rigid, and hence the whole surface is rigid

(3) In the first experiment it was found that the contact of three pairs chosen from AA_1, BB_1, CC_1, DD_1, involved that of the fourth, provided the fourth was not CC_1. We discover that while A, B, D remain in contact with the corresponding points, an initial contact of C with C_1 may be interrupted. We then declare that A, B, C, D and A_1, B_1, C_1, D_1 have moved with respect to each other, and during the motion three pairs of points have remained in contact. This is the characteristic of a straight line. A, B, D lie on a straight line and so do A_1, B_1, D_1. A straight line is thus defined by point-pairs that remain fixed with respect to a rigid frame during a rotation about the line.

(4) If we bring A into contact with A_1 and simultaneously B in contact with B_1', B_1'', $\cdots$ one after another, it never occurs that D is not in contact with a point of the surface D_1', D_1'', $\cdots$. The points A_1, B_1', D_1', lie on a straight line, also A_1, B_1'', D_1'', and so forth.

(5) If the preceding experiment is performed with A in contact with A_2, A_3, etc., the same results are obtained. Thus from every point in the surface there extend straight lines in the surface in all directions, and hence the surface is judged to be a plane.

As a result of the preceding experiments we have learned how to recognize a straight line and a plane. In practice we test the straightness of a line by the physical law that light travels through a homogeneous medium in straight lines. Straight lines are exemplified by the edge of a solid, by a stretched cord, and by the path of a ray of light.

Our next task is to introduce the concept of distance or length. Suppose that we have given two stretches determined by rigid point-pairs (A,B) and (A_1,B_1) respectively, so that A is in contact with A_1 and B is in

contact with B_1. As previously stated, the stretches are said to be congruent. The same length, or distance between their end points, is assigned to each of the congruent stretches. Congruence is directly tested when corresponding points are in contact, but this test fails when the stretches are separated. However, we shall assign the same length, or distance, to the separated stretches. We thus adopt the fundamental postulate that the length of a stretch determined by a rigid point-pair, or the distance between the two points, is invariant in displacement. It is assumed, however, that the temperature remains constant. A standard stretch may be assigned the length one. The length of any straight line can then be determined with respect to our standard. We may measure the length of a line by counting the number of times that the standard can be laid off on the line, or by counting the number of equal stretches that may be placed end to end along the line.

The operational significance of the concept of length is especially exemplified by the special theory of relativity. I have already stated that space is associated with some frame of reference. A fundamental assumption of classical kinematics was that space is absolute, that is, the same for all frames of reference regardless of their state of motion. This means that the geometrical properties of figures were viewed as invariant under a transformation of the frame of reference. Thus the length of a rigid rod was postulated to be the same relative to frames of reference in relative motion with respect to one another. Indeed, it appears to be self evident that the length of a rod represents an intrinsic property which does not depend on the frame of reference. According to the operational theory, however, the concept of length is defined by the method of measurement, and in relativistic theory the result depends on the state of motion of the frame. If the frame is one in which the rod is at rest, an observer can measure the length of the rod in terms of a standard of length by placing a calibrated scale of length adjacent to the rod under investigation and observing the points on the scale that coincide with the end points of the rod. But in a frame relative to which the rod is moving, this procedure is not possible because of relative motion between the rod and the instrument of measurement. A possible procedure is to mark the simultaneous positions of the end points of the rod on the frame of reference. One may then at one's leisure measure the distance between the two points on the frame of reference with a scale at rest. Simultaneity, however, is relative to the frame of reference, and hence the outcome of measuring length is relative. In general, in the theory of relativity the geometrical structure of a body is relative to the frame. A configuration which is described as a circle from a frame relative to which it is at rest is described as an ellipse from a frame relative to which it is moving.

Let us now return to the problem of constructing a physical geometry for structures at rest in a selected frame of reference. We have a standard of length and methods for the recognition of straight lines and planes. We may verify the proposition that a straight line is shorter than an adjacent line between the same points; this proposition may be used to define a straight line. We may construct figures out of straight lines. The properties of a plane triangle may be used to determine the curvature of the plane; the curvature is zero, negative, or positive according as the sum of the angles is equal to, less than, or greater than two right angles. The curvature of three mutually perpendicular planes at a point determines the curvature of space at that point.

By such procedures we build up a concept of metrical physical space. The positional relations of rigid bodies which determine the metrical structure of space are described by a geometry which is a branch of physics. Applied to the physical world of experience, our procedures yield the result that to the first approximation, at least, actual physical space is Euclidean. The sum of the angles of large triangles, the sides of which are the paths of light rays, is two right angles. It is possible to construct a Cartesian coördinate system out of equal rods. This means that out of a set of rods, the corresponding end points of which coincide when the rods are placed adjacent to one another, it is possible to construct a cubical lattice which is the physical realization of a Cartesian coördinate system.

The propositions that characterize the positional properties of configurations of rigid bodies are only approximately verified by experience on account of lack of precision in observation. In the development of geometry, the fiction of a precise observation is adopted and the propositions are interpreted to express definite relations between definite properties. This procedure makes it possible to study the deductive relations between propositions, and Euclidean geometry may then be founded on axioms which express the properties of a set of terms and relations. We may then transform these axioms into a set of postulates which implicitly define the formal properties of the objects of geometry and thereby obtain an abstract geometry. The structures of physical geometry then exemplify approximately the formal properties defined by the postulates. *In the passage from physical to abstract geometry it does not appear to be necessary to interpolate a science that is founded on pure intuition.*

4. Analytic treatment. In the foregoing discussion I have employed the synthetic method of building geometry, but one may use the analytic method. A Euclidean space is characterized by the fact that it admits a

cubical lattice which will serve as a Cartesian coördinate system. The metrical structure of space is described by the formula which expresses the differential element of distance between two points in terms of the differences in the coördinates of the points. Thus Euclidean space admits a Cartesian coördinate system for which

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Curvilinear coördinates may also be used, but the formula for the line element in such coordinates can always be transformed to the Cartesian form.

On a curved surface it is impossible to extend a Cartesian coördinate system over a finite region. Accordingly one introduces Gaussian, that is, curvilinear, coördinates. The coördinate lines may be labelled $u_1 =$ constant and $u_2 =$ constant. The position of a point on the surface is specified by giving its Gaussian coördinates u_1 and u_2. The distance between two points u_1, u_2 and $u_1 + du_1, u_2 + du_2$ is expressed by the formula

$$ds^2 = g_{11}du_1^2 + 2g_{12}du_1du_2 + g_{22}du_2^2.$$

The g's are function of u_1 and u_2 and are called the components of the fundamental metrical tensor. The measure of curvature of the surface is a function of the g's and their derivatives.

In the classical accounts of differential geometry the curved surface is viewed as imbedded in a three-dimensional Euclidean space. The g's are expressed as functions of the derivatives of the Cartesian coördinates with respect to the Gaussian coördinates on the surface u_1, u_2,

$$g_{ik} = \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_k} + \frac{\partial y}{\partial u_i}\frac{\partial y}{\partial u_k} + \frac{\partial z}{\partial u_i}\frac{\partial z}{\partial u_k}.$$

A physicist, however, prefers an exposition of the immediate physical significance of the g's. The discussion presupposes that in an infinitesimal region the surface may be assumed plane. I assume that we have a standard of length which is invariant during displacements on the surface. ds is the length of the element of arc between the two points relative to the standard. du_1 and du_2 are increments of coördinates and have no immediate metrical significance. As we pass from u_1, u_2 to $u_1 + du_1$, u_2, the distance ds is related to the coördinate increment du_1 by $ds^2 = g_{11}du_1^2$. Then $ds = \sqrt{g_{11}}\, du_1$. Thus $\sqrt{g_{11}}$ is the ratio of distance advanced to increment in coördinate u_1. For example, if $ds = \frac{1}{2}$ for $du_1 = 1$, $\sqrt{g_{11}} = \frac{1}{2}$. This means that if a unit of length is placed on the coördi-

nate line $u_2 =$ constant, one half of the unit extends from the line u_1 to $u_1 + 1$. A similar explanation is given for $\sqrt{g_{22}}$. If θ is the angle between the coördinate lines, $g_{12} = g_{21} = \cos\theta \sqrt{g_{11}}\sqrt{g_{22}}$. The metrical structure of the surface is known if we determine the g's for every point of the surface. Let us now consider the application of the methods of analytic geometry to physics.

5. A METRIC FOR A GEOMETRY FOR PHYSICS. Classical physics was founded on the assumption that physical space is Euclidean. This means that a set of equal rigid rods can be fitted together to form a cubical lattice of finite extent. The lines of the lattice may be used as the lines of a Cartesian coördinate system. Cartesian coördinates directly express the distance of a point from a coördinate plane and hence have direct physical significance. If Cartesian coördinates are symbolized by x_1, x_2, x_3, the the metric of Euclidean space is expressed by the formula $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$. To the first approximation, at least, physical space is Euclidean, and this fact explains the universal application of Euclidean geometry in classical mechanics.

The special theory of relativity provided a basis for a four-dimensional space-time relational structure of events. In addition to three spatial coördinates x_1, x_2, x_3 there was introduced a fourth coördinate, the value of which is directly related to the time indicated by a clock. If t is time indicated by a clock, we may define $x_4 = ict$. Then

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

expresses the metric of space-time. ds, the invariant interval between two events, is thus expressed in terms of differences of spatial coördinates and the time.

The general theory of relativity assumes that space-time is a continuum characterized by a Riemannian metric. In a gravitational field the positional relations of rigid bodies do not satisfy the propositions of Euclidean geometry. It is not possible to build finite Cartesian lattices out of equal rigid rods. The rate of clocks is affected by a gravitational field. Hence the metrical structure of a space-time region containing a gravitational field cannot be expressed by the formula for ds used in the special theory. The more general Riemannian formula

$$ds^2 = \Sigma\, g_{ik} dx_i dx_k$$

is necessary. The g_{ik} have a physical significance that may be defined by a procedure similar to the one for the two-dimensional surface.

The theory that physical space-time is Riemannian raises the problem of how the standard of measure for ds is set up at a particular space-time point. In ordinary space this is accomplished by bringing a standard of length to the point. But the interval of space-time contains spatial and temporal factors. The metrical evaluation of ds may be made with the aid of the special theory of relativity. In a relatively small space-time region it is possible to select a frame of reference relative to which there is no gravitational field. A gravitational field is relative to a frame of reference and will vanish relative to a suitable accelerated frame. For example, there is no gravitational field relative to an elevator which is falling freely towards the surface of the earth. Relative to the frame with respect to which there is no field and in which the coördinates of an event are x_1, x_2, x_3, x_4, the interval between two events may be expressed by

$$ds^2 = dx_1^2 + ds_2^2 + dx_3^2 + dx_4^2;$$

and dx_1, dx_2, dx_3, dx_4 may be determined by rigid rods and clocks as in special relativity, and hence the value of the corresponding ds can be calculated.

The geometrical significance of the g_{ik} is part of their physical significance. The g_{ik} also have a dynamical significance, for they are the potentials of the gravitational field. The law of gravitation expresses a condition on the g's and their derivatives. The fundamental law of motion is that a free particle describes a geodesic in curved space-time. In this sense physics is reduced to geometry, but geometry is a branch of physics.

6. Summary. This paper may be summarized by a restatement of the relation between physical geometry and abstract geometry. Typical propositions of Euclidean geometry may be formulated as generalizations from experiences of practically rigid bodies. Such laws are expressed in terms of quantities which may be determined within limits of precision. The next step is to assume that the propositions hold exactly for a set of objects, such as ideal rigid bodies. Propositions with a precisely defined content may be reduced to a set of axioms from which theorems can be deduced. The status in reality of ideal objects is uncertain. Historically the attempt has been made to give them reality in a transcendent realm or to view them as constructions in pure intuition. The problem of the ontological status of the objects of geometry is avoided by eliminating the empirical reference of the concepts. The axioms then become postulates which implicitly define the formal properties of the objects of the concepts. Thus generalizations from experience become transformed into definitions. The self-evidence which has

been attributed to the axioms of Euclidean geometry is founded on their status as definitions. The proposition that a straight line is the shortest distance between two points is self-evident in the sense that it may be used as the definition of a straight line.

Once we have the concept of abstract geometry, it is possible to create new abstract geometries and then seek physical interpretations of them. The interest in differential geometry stimulated by the general theory of relativity has resulted in the invention of non-Riemannian geometries. The geometry of Weyl, for example, is based upon the assumption that the standard of distance is a function of position. But such developments are beyond the scope of this paper.

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FOREWORD

The word "dimension", not unlike a goodly number of other familiar mathematical terms, is often used loosely and ambiguously by the layman. Just as the words "countless" and "infinite" are frequently used to convey the idea of a very great many objects, so the word "dimension" is frequently associated with the notion of magnitude or size. Thus a rectangle is said to be two-dimensional because its area is given by the product of its two dimensions — its *length* and its *width*. Similarly, a rectangular prism is said to be three-dimensional because its volume is given by the product of its three dimensions: *length, width* and *height*. Such a conception of dimension is quite naive, if not almost meaningless, mathematically.

A mathematically meaningful approach to the concept of dimensionality is through the notion of simplexes, or what might be called "basic" sets of points. Thus a simplex of dimension zero is a single point. A simplex of dimension 1 is a line segment joining two vertices (its faces). A simplex of dimension 2 is a triangle together with its interior; its 1-dimensional faces are its three sides, and its 0-dimensional faces are its three vertices. A simplex of dimension 3 is a tetrahedron together with its interior; its 2-dimensional faces are its four triangular "surfaces", its 1-dimensional faces are its six "edges", and its 0-dimensional faces are its four vertices.

In the present essay, still another approach is taken to the concept of dimension, namely, one involving the notions of *neighborhood* and *boundary*. Although these two notions are familiar and intuitively easily comprehended, they nevertheless lead to a sophisticated abstract point of view which is characteristic of contemporary mathematics, and which clarifies much of the thinking about the relation of mathematics to physical reality.

Since the reader may not be familiar with the terminology of topology, a few remarks may prove helpful.

A set E of points is said to be *compact* (1) if E contains only a finite number of points, or (2) if every infinite subset of points of E has at least one accumulation point in E. By an *accumulation point* of a set of points

is meant, somewhat loosely, a point P which is the limit of a sequence of points of the set.

A set of points is said to be a *connected set* if it cannot be divided into two sets R and S which have no points in common and which are such that no accumulation point of R belongs to S and no accumulation point of S belongs to R. A *simply connected* set is a set such that each pair of its points can be joined by a simple arc all of whose points are in the set. A *simple arc* is a set of points which can be put into one-to-one correspondence with the points of the closed interval [0, 1] in such a way that the correspondence is continuous in both directions. Roughly, this means a segment of a curve that does not "cross itself". All simply-connected sets are connected; but not all connected sets are simply-connected.

These observations may help the reader to understand more of Professor Menger's essay; on the other hand, if the technical subtleties still seem a bit formidable, he should not be dismayed. Instead, by reading and re-reading, it is altogether likely that some insight into these rather significant concepts will be gained nevertheless.

WHAT IS DIMENSION?

KARL MENGER

1. SOLIDS, SURFACES, AND LINES. Strictly speaking, all material objects are 3-dimensional. Yet, only such objects as a metal sphere, a wooden block, or a rock are considered to be typical representatives of 3-dimensional entities (solids). A piece of sheet-iron, paper, and a membrane approach what we mean when we speak of 2-dimensional objects (surfaces). Wire, threads, and streaks of chalk represent our idea of 1-dimensional entities (lines).

What is the difference between objects of different dimensions? Originally, mathematicians believed it to be a difference in quantity, in the sense that a surface contains more points than a line and less points than a solid. Now primarily the words "more," "less," and "equally many" are restricted to finite sets while surfaces, as well as lines and solids, contain infinitely many points. But Georg Cantor extended their use to all sets. We say that two sets—finite or infinite—contain equally many elements if we can establish a one-to-one correspondence between their elements. Cantor found that two infinite totalities do not necessarily contain equally many elements. For instance, among geometrical objects a straight line segment contains more points than some dispersed infinite sets, *e.g.*, the set of all points on a straight line whose distances from a certain point are integers. However, a straight line segment, a square, and a cube do contain equally many points [1]. Since these objects are of different dimensions, it follows that dimension is not a quantitative property.

Later, geometers throught that the difference between a 1-dimensional and a higher-dimensional object lay in the fact that the former, but not the latter, can be traversed by a continuously moving point. Indeed, lines on a paper or a blackboard are drawn, *i.e.*, traversed by the point of a pencil or chalk. However, Peano found that a continuously moving point can traverse a square surface or a solid cube though nobody would call these objects 1-dimensional. On the other hand, 1-dimensional objects were found which cannot be traversed by a continuously moving point [2]. The fact that an object is the path of a point is interesting in itself, but has no bearing on the question of the dimension of the object [3].

When one-to-one, as well as continuous, mappings had proved to be inadequate bases for the definition of dimension, mathematicians attempted to characterize dimension of a totality T as the least number of real numbers required to describe topologically (in a one-to-one and bicontinuous way) the elements of T. Each point of our ordinary space can be topologically characterized by three, but not less than three, real numbers, *e.g.*, its Cartesian or spherical coordinates; each point of a simple surface by two, but not less than two, real numbers, *e.g.*, the points of a sphere by longitude and latitude; each point of a simple line by one number. Thus, by the last definition, our space is 3-dimensional, simple surfaces are 2-dimensional, simple lines are 1-dimensional. Similarly, each color sensation of a normal eye can be topologically characterized by three, but not less than three, real numbers, *viz.*, the quantities of three standard colors whose mixture produces an identical sensation. Hence, the totality of color sensations of a normal eye is 3-dimensional while the corresponding totalities for a partially or totally color blind eye are but 2- and 1-dimensional, respectively. In the same way, a totality of all mixtures of four ingredients which cannot be obtained by mixing less than four of them is called four-dimensional. In fact, in this direction lies our only elementary analytical approach to the fourth dimension and higher-dimensional spaces.

Unfortunately, however, the last definition applies only to very simple spatial entities, *viz.*, to those which can be obtained by means of a very simple transformation from a straight segment, a square, or a cube. Such entities are called arcs, discs, and topological spheres. In our space and in the plane, arcs and discs form only a small part of the lines and surfaces studied by modern geometry. Even if we admit objects which are sums of a finite number of arcs and discs our domain is still very restricted. For instance, the line mentioned above, which cannot be traversed by a continuously moving point [2], does not belong to this domain since it is not a sum of a finite number of arcs. In fact, it is the sum of infinitely many arcs, but all sets which are sums of infinitely many arcs cannot possibly be called 1-dimensional since the square and the cube are sums of infinitely many straight segments [4].

To formulate the intuitive difference between lines, surfaces, and solids one can devise a simple experiment whose outcome depends upon the dimension of the object to which it is applied [5]. We cut out from the object a piece surrounding a given point. If the object is a solid we need a saw to accomplish this, and the cutting is along surfaces. If the object is a surface a pair of scissors suffices, and the cuts are along curves. If we deal with a curve we may use a pair of pliers and have to pinch the

object in dispersed points. Finally, in a dispersed object no tool is required to perform our experiment, since nothing needs to be dissected. This characterization of dimension leads from n-dimensional to $(n - 1)$-dimensional objects. It ends with dispersed sets, naturally called 0-dimensional, and, beyond these, with "nothing," in set theory called the "vacuous set." It is, therefore, convenient to consider the latter as -1-dimensional.

2. THE DEFINITION OF DIMENSION. To make this idea precise we need only two simple auxiliary concepts: neighborhood and boundary. In our space we call a set N a *neighborhood* if each point of N is center of a sphere (though perhaps a very small sphere) all of whose points belong to N. The interior of a cube is a neighborhood, whereas a cube with its faces is not. For even the smallest sphere about a point of a face contains points not belonging to the cube. Nor is a plane a neighborhood in our space. For each sphere about each point of a plane contains points not belonging to the plane. The *boundary* of a neighborhood N is the set of all points which do not belong to N but are centers of arbitrarily small spheres which contain some points of N. For the interior of the cube the boundary obviously consists just of the six faces.

In terms of these concepts the result of our recursive dimension experiment can be explained as follows: A set S of points of our space is *at most n-dimensional* if each point of S lies in arbitrarily small neighborhoods whose boundaries have at most $(n - 1)$-dimensional intersections with S. The set S is *n-dimensional* if it is at most n-dimensional but not at most $(n - 1)$-dimensional. That S is not at most $(n - 1)$-dimensional means that S contains at least one point at which S is at least n-dimensional, that is to say, a point which does not lie in arbitrarily small neighborhoods whose boundaries have at most $(n - 2)$-dimensional intersections with S; the boundaries of all sufficiently small neighborhoods of such a point have at least $(n - 1)$-dimensional intersections with S. The vacuous set called -1-dimensional, is the starting point of the recursive definition [6].

By this definition, a set S is 0-dimensional if it is not vacuous, and each point of S lies in arbitrarily small neighborhoods whose boundaries have -1-dimensional, *i.e.*, vacuous, intersections with S — in other words, no points in common with S. A set S is 1-dimensional if it is not 0-dimensional and each point lies in arbitrarily small neighborhoods whose boundaries have at most 0-dimensional intersections with S. But it should be clearly understood that a point of a 1-dimensional set S may also be contained in arbitrarily small neighborhoods whose boundaries have more than 0-dimensional intersections with S. For instance, each point

of a straight line S is contained in arbitrarily small neighborhoods whose boundaries contain whole pieces of S. Such neighborhoods can be formed by adding two cubes of different size, one of which has a face passing through S.

Furthermore, it should be clear that we cannot always expect to find *simple* neighborhoods of a point of an n-dimensional set whose boundaries have at most $(n - 1)$-dimensional intersections with S. One of the most interesting examples in this respect arises from the study of the following four sets whose sum incidentally exhausts our space:

the set S_0 of all points which have three irrational coordinates,

the set S_1 of all points which have one rational and two irrational coordinates,

the set S_2 of all points which have two rational and one irrational coordinate,

the set S_3 of all points which have three rational coordinates.

If a, b, c are any three rational constants, then the planes $x = a, y = b, z = c$ do not contain any points of S_0, and the planes $ax + by + cz = 1$ do not contain any points of S_2. If α, β, γ are any three irrational constants, then the planes $x = \alpha, y = \beta, z = \gamma$ do not contain any points of S_3. Now, for $i = 0, 2, 3$, each point of S_i is contained in arbitrarily small *cubes* whose faces are part of such planes which have no point in common with S_i. Hence, S_0, S_2, S_3 are 0-dimensional. So is S_1 but the proof of this fact is much more difficult [7]. For not only each plane meets S_1, but as Schreier noticed, each surface of the form $z = f(x, y)$ where f is a continuous function, has points in common with S_1 and the same is true for each surface $y = f(x, z)$ and $x = f(y, z)$. In fact, only recently S. G. Reed, Jr. and the author constructed [8] a neighborhood whose necessarily complicated boundary has no point in common with S_1.

Since dimension of the subsets of our space has been defined in terms of neighborhoods the definition is applicable to the subsets of all spaces in which neighborhoods are given. An example of such a space is the 4-dimensional euclidean space whose points are the quadruples of real numbers $x, y, z\ u$ and in which the sphere with radius r and center x_0, y_0, z_0, u_0 consists of the points x, y, z, u satisfying the inequality $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + (u - u_0)^2 \leq r^2$. A set N is a neighborhood if each point of N is center of a sphere all of whose points belong to N.

3. CRITERIA FOR A SATISFACTORY DEFINITION. Now let us examine the definition of dimension. Its objective is to make precise and to extend the

ordinary usage of the words "1-dimensional," "2-dimensional," and "3-dimensional." *A good definition of a word must include all entities which are always denoted and must exclude all entities which are never denoted by the word.* For the word "1-dimensional" straight lines, ellipses and lemniscates are objects of the former type; square surfaces, solid cubes, and finite sets of the latter type. *A good definition should extend the use of the word by dealing with objects not known or not dealt with in ordinary language. With regard to such entities, a definition cannot help being arbitrary.* In connection with the word "1-dimensional" consider the four sets S_i whose sum exhausts our space. A general definition of "1-dimensional" will imply for each of the sets S_0, $S_0 + S_1$, $S_0 + S_1 + S_2$ whether or not it is 1-dimensional. Our definition implicitly assigns to these sets the dimensions 0, 1, and 2, respectively, which like each assignment is somewhat arbitrary since ordinary language does not assign to them any dimension. *A good definition must yield many consequences, in particular theorems which are aesthetically satisfactory by their generality and simplicity, and theorems connecting the defined concept with concepts of other theories.* It is these theorems which justify the unavoidable arbitrary element of the definition. Some of the theorems will extend statements which are true in the restricted domain of ordinary language to the extended domain of the definition. Other theorems will exhibit interesting exceptions or even correct erroneous habits of thinking.

The definition outlined in this paper has yielded an extensive dimension theory which, since its foundation in the early twenties, has developed into one of the central branches of topology. Since even an enumeration of the main results would surpass the limits of this paper we shall confine ourselves to a few illustrations of the general criteria of this previous paragraph. An example of the numerous statements extending to all sets a proposition known to hold for the simple objects of ordinary language, is the theorem [5] that an n-dimensional set S contains infinitely many points at which S is n-dimensional, and that these points form a set S' which is at least $(n-1)$-dimensional. Under certain conditions we can say that S' is n-dimensional. However, there are rather unexpected exceptions in which S' is only $(n-1)$-dimensional. One of the facts which justify our definition of 0-dimensionality is the simple and beautiful general theorem that each n-dimensional set is the sum of $n+1$ but not less than $n+1$ 0-dimensional sets. If we had assigned to the sets S_0, S_0+S_1, $S_0+S_1+S_2$ other dimensions than we did, it would have been at the expense of a simple systematic theory.

4. Five properties of dimension. In concluding, I shall select five of the theorems of dimension theory which, as we shall see, are of a particular importance:

I. *The euclidean n-space is n-dimensional.* (This theorem is due to Brouwer.) The cases $n=1, 2, 3$ of this theorem show, in particular, that the definition of 1-dimensionality excludes square surfaces and solid cubes which ordinary language always excludes and which older definitions failed to exclude.

II. *The topological image of an n-dimensional set is n-dimensional.* In conjunction with theorem I this simple theorem shows that the concepts of 1-dimensional and 2-dimensional sets include arcs and discs which are always called 1-dimensional and 2-dimensional, respectively.

III. *Each part of an n-dimensional set is at most n-dimensional.* Natural and simple as this theorem is it does not hold for some other definitions of dimension [9].

IV. *A set S cannot be split into denumerably many* [4] *closed* [10] *summands each of which is of smaller dimension than S.* (This so-called sum-theorem which occupies a central role in dimension theory, as well as the simple theorems II and III are due to Urysohn and the author.)

V. *Each n-dimensional set can be topologically transformed into a subset of a compact* [10] *n-dimensional set.* (This theorem is due to Hurewicz.)

5. FURTHER ASPECTS OF THE PROBLEM. What is dimension? Have we answered this question? In one sense, we have. We have explained which sets are 1-dimensional, which are 2-dimensional, etc. In fact, with each subset of our space and with each subset of much more general spaces we have associated an integer, the dimension of the set. This is also expressed by saying that dimension is a set function. However, there are many other set functions. With each set in our space we may, for example, associate the number of pieces of which it consists, or its measure (in some sense). In this connection the question "What is dimension?" may be interpreted in the following sense: "Among the many set functions, by which properties is dimension characterized?"

So far this question has only been answered for the plane [11]. There dimension is characterized by the properties described in theorems I to V, that is to say: In the plane, dimension is the only set function with the following properties:

1) It assumes the values 2, 1, 0, —1 for the square, the straight line segment, the single point, and the vacuous set, respectively;

2) It assumes the same value for any two sets which can be obtained from each other by a topological transformation;

3) It never has a greater value for the part than for the whole;

4) No set can be split into denumerably many closed sets of smaller function value;

5) Each set can be topologically transformed into a part of a compact set of equal function value.

In the plane, therefore, this is another answer to the question, "What is dimension?"

Footnotes

[1] Also some dispersed sets and a straight line segment contain equally many points, *e.g.*, the set of all points on a line whose distance from a certain point is irrational, or Cantor's so-called *discontinuum*.

[2] *E.g.*, the so-called *sinusoid* consisting of the points (x, y) of the plane for which either $0 < x \leq 1$ and $y = \sin 1/x$ or $x = 0$ and $-1 \leq y \leq 1$.

[3] In this connection we may mention a comparatively recent result of the theory of curves. If a set S as well as each subcontinuum of S can be traversed by a continuously moving point, then S is 1-dimensional in the sense defined in this paper. The converse of this theorem is not true.

[4] One might think that 1-dimensional are the sets which are the sum of *denumerably many* arcs, *i.e.*, of as many arcs as there are integers. But this definition would still be too narrow while the class of entities which are sums of *non*-denumerably many arcs contains the square and the cube and thus is too wide.

[5] See the author's book "Dimensionstheorie" 1928.

[6] The history of this definition and the ensuing theory is outlined in the beautiful exposition of Hurewicz and Wallman, Dimension Theory, Princeton University Press, 1941.

[7] See Hurewicz and Wallman, p. 19.

[8] To be published in Issue 5 of the Reports of a Mathematical Colloquium, University of Notre Dame publication.

[9] See the Appendix to Hurewicz and Wallman, Dimension Theory.

[10] A set C is *closed* if its complement is a neighborhood, and hence C contains all cluster points of C, *i.e.*, all points of which each neighborhood has infinitely many points in common with C. A set C is called *compact* if for each infinite subset of C there exists a cluster point in C. It should be noted that theorem IV would not hold if we omitted the word closed: our 3-dimensional space can be split into a finite number of sets of smaller dimensions which are not closed, *e.g.*, into the four 0-dimensional sets S_0, S_1, S_2, S_3. Nor would theorem IV hold if we admitted splitting in more than denumerably many closed sets. Our 3-dimensional space is sum of infinitely many (but not denumerably many) closed 0-dimensional sets, *e.g.*, of sets each of which consists of exactly one point.

[11] Monatshefte f. Mathematik u. Physik, 36, 1929, p. 193.

FOREWORD

More than half a century ago, in his classical book *Elementary Mathematics from an Advanced Standpoint,* Felix Klein observed that "everyone knows what a curve is, until he has studied enough mathematics to become confused through the countless number of possible exceptions." Do not be lightly deceived by simple words in mathematics, such as *straight, flat, round, curve,* and so on. The late Professor Kasner once gave a "booby prize" definition of mathematics as "the science which uses easy words for hard ideas." Thus what a mathematician calls a "simple curve" might look very complex to you, while a curve such as the figure "8" is not considered a simple curve, however "simple" it may appear.

Thus we encounter all sorts of curves in mathematics: continuous curves; curves with "breaks" or discontinuities; curves with special points; curves which have tangents and curves which do not; and so on. Then there are so-called *pathological curves,* as for example, the Snowflake curve, which, although infinite in length, can nevertheless be drawn on a small finite area, say a visiting card; the In-and-Out Curve, whose curvature cannot be measured; the Space-filling Curve, which, when complete, passes through every point in a given square (or cube); and, hard to believe, the Cisscross Curve, which crosses itself at every one of its points.

After perusing the following article, the reader may wish to learn more about mathematical curves. To this end, he will find the following references of interest:

1. Edward Kasner and James Newman, *Mathematics and the Imagination,* pp. 343–356.
2. Oystein Ore, *Graphs and Their Uses,* pp. 5–20; 53–67.
3. Hans Rademacher and Otto Toeplitz, *The Enjoyment of Mathematics,* pp. 61–66; 163–177.
4. Hugo Steinhaus, *Mathematical Snapshots,* pp. 46–61; 95–108.

WHAT IS A CURVE?

G. T. WHYBURN

1. INTRODUCTION. When the searching light of modern mathematical thinking is focused on the classical notion of a curve, this idea is found to involve elements of vagueness which must be clarified by accurate and exact definition. Fortunately this has been made possible and relatively simple by development in the field of set-theoretic topology. We shall endeavor to set forth below, first the need for explicit definition of a curve, then the definition itself, and finally several illustrations of types of simple curves which can be completely characterized by their topological properties and which more nearly approach the classical notion of a curve.

2. THE CLASSICAL NOTION. The concept of a curve as the "path (or locus) of a continuously moving point" usually is accompanied by intuitive notions of *thinness* and *two-sidedness*. When the curve is in a plane, these were thought to be consequences of the rather vaguely formulated definition of a curve as just given.

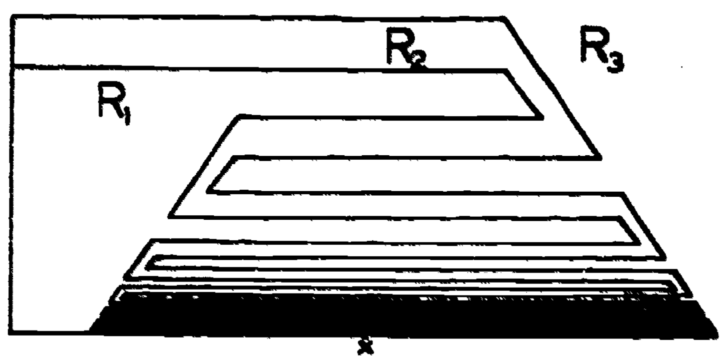


FIG. 1.

That the path of a continuously moving point is not necessarily a thin or curve-like set was shown by Peano and somewhat later by E. H. Moore, who demonstrated the remarkable fact that a square plus its interior can be exhibited as the continuous image of the interval. In other words, if S denotes a square plus its interior, we can define continuous functions $x(t)$ and $y(t)$ on the interval $0 \leq t \leq 1$ so that as t varies from 0 to 1, the point $P[x(t), y(t)]$ moves continuously through all the points of S.

A still more striking result in this direction is the remarkable theorem proved independently by Hahn and Mazurkiewicz about 1913. This theorem asserts that in order for a point set M (in euclidean space of any number of dimensions) to be representable as the continuous image of the interval $0 \leqq t \leqq 1$, it is necessary and sufficient that M be a locally connected continuum. (A *continuum* in euclidean space is a closed, bounded, and connected set; and a continuum M is *locally connected* provided that for any $\varepsilon > 0$ a $\delta > 0$ exists such that any two points x and y of M at a distance apart $< \delta$ can be joined by a subcontinuum of M of diameter $< \varepsilon$). Thus since obviously not only a square but also a cube, an n-dimensional interval, an n-dimensional sphere and a multitude of other sets are locally connected continua, any such set M can be represented as the path of a continuously moving point in the sense that we can define continuous functions.

$$x_i = x_i(t) \qquad 0 \leqq t \leqq 1, i = 1, 2, \cdots, n,$$

such that as t varies from 0 to 1 the point P with coordinates $(x_1, x_2, \cdots, x_n)$ moves continuously through all the points of M.

Even when a set is sufficiently "thin" or "1-dimensional" that we would probably call it a curve it may be in a plane and still not be two-sided. To illustrate we note that in Figure 1 any point on the base of the continuum, such as x, is a boundary point of each of the three regions R_1, R_2, R_3 into which the continuum divides the plane. Hence there are *three sides* of the base of this continuum. (Clearly we could add extra oscillating curves to the figure so as to make an arbitrarily large number or even an infinite number of regions each having all base points x on their boundaries). Nevertheless our continuum is a thin 1-dimensional set made up of an infinite number of line segments. Now it is possible to construct in a plane a continuum which is thin in the sense that it will not contain the interior of any circle and yet is so unusual that it will divide the plane into any finite number or an infinite number of regions and, further, it will be the boundary of each one of these regions. Also a plane continuum can be constructed which not only itself cuts the plane into infinitely many regions but has the remarkable property that every subcontinuum of it (any "piece" of it) also cuts the plane into infinitely many regions.

3. Dimensionality. General definitions of curve, surface, solid. Undoubtedly sufficient evidence has been given of the necessity of being precise in our definitions and statements concerning curves, surfaces, etc., and of the unreliability of our intuition concerning these concepts.

We leave aside the continuous traversibility of the set as a criterion

characterizing or distinguishing between curves, surfaces, solids, etc., since we have seen how it fails in this respect, and concentrate on content or dimensionality of the set as a guide.

Hence it seems natural and adequate to define a *curve* as a 1-dimensional continuum, a *surface* as a 2-dimensional continuum and a *solid body* as a 3-dimensional continuum.

These definitions are satisfactory provided we give an adequate definition of dimensionality of a set. To this end let us concentrate our attention on compact sets, *i.e.*, sets K which have the property that any infinite subset has a limit point belonging to K, sets which are closed and bounded if they lie in a euclidean space.

We then define the dimensionality of the empty set to be -1 and agree the dimensionality of any other set is to be $\geqq 0$. Assuming, then, that we have defined the dimensionality concept for dimensions $\leqq n-1$, by induction we define a set K to be of dimensionality n provided (1) every pair of distinct points p and q of K can be separated in K by some set X of dimensionality $\leqq n-1$, *i.e.*, $K-X$ falls into two separated sets K_p and K_q containing p and q respectively; and (2) some pair of points of K cannot be separated in K by a subset of K of dimensionality $< n-1$. Thus for $n \geqq 0$, a set K is of dimension n provided n is the least integer such that every pair of distinct points of K can be separated in K by the removal of a subset of dimension not greater than $n-1$.

According to this definition, then, a compact set K is of dimension 0 provided every two points of K can be separated in K by omitting the empty set, *i.e.*, provided they are already separated in K. Hence a 0-dimensional set is one which is non-empty but is totally disconnected in the sense that its only connected subsets are single points. A compact set K is 1-dimensional provided any two points can be separated in K by omitting from K a 0-dimensional or totally disconnected set but some two points cannot be separated without omitting some points from K. A compact set K is 2-dimensional provided each pair of points of K can be separated in K by omitting a 1-dimensional set but not every pair can be separated by omitting a 0-dimensional set, and so on.

Stated in other terms, if we accept our definition that a curve is a 1-dimensional continuum, a surface is a 2-dimensional continuum, and a solid body is a 3-dimensional continuum, we see that a non-empty compact set K is 0-dimensional if every pair of its points are separated in K. The set is 1-dimensional at most provided we can (with shears if you like) separate any two of its points by cutting the set along a 0-dimensional set, *i.e.*, by cutting out only single points as connected pieces. The set is

2-dimensional at most provided we can separate any two points by cutting the set along a 1-dimensional set, *i.e.*, by cutting out only curves as connected sets. The set is 3-dimensional at most if we can separate any two points by cutting (with a saw perhaps) the set along a 2-dimensional set, *i.e.*, by cutting out only surfaces as connected sets.

4. SOME SIMPLE TYPES OF CURVES. Having defined exactly the notions of curve, surface, and solid in terms of their topological properties in such a way that they correspond roughly to the geometrical notions of line, plane, and space, we consider now some interesting particular kinds of curves which may be similarly characterized.

Take first a straight line interval ab joining two points a and b and ask the question "What properties of a set make it essentially like an interval?" or "When are the points in a set associated together like those in the interval ab?" For example, if ab is a taut string and we release the tension and let it go slack but do not allow it to loop over onto itself, it is no longer straight but it retains its same essential structure. It can still be severed by cutting out any one of its points other than a or b; and it is this property in particular which characterizes the interval completely from the topological point of view. In other words, if we understand by a *simple arc* any set of points which is topologically equivalent to an interval in the sense that its points can be put into one-to-one and continuous correspondence with the points of an interval, then *in order that a continuum T be a simple arc it is necessary and sufficient that T contain two points a and b such that the removal of any point of T other than a or b will disconnect T.* Thus in Fig. 2, (a) is a simple arc, but (b) is not a simple arc because the removal of neither a, b, nor x will separate the the set (*i.e.*, will make it fall apart).

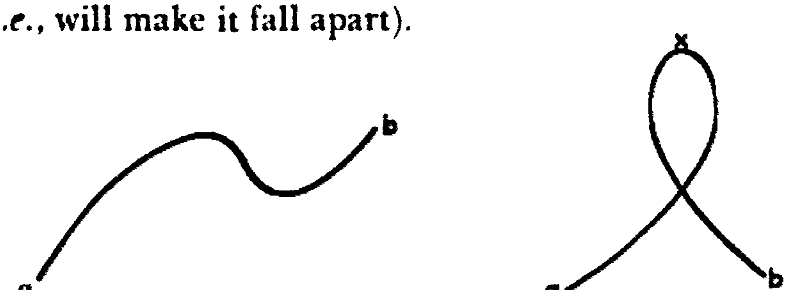
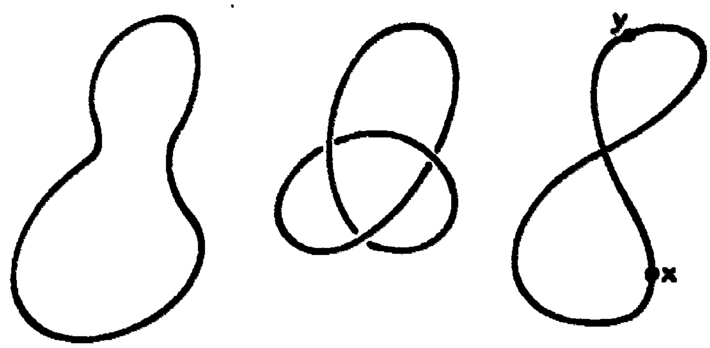


FIG. 2

Consider next a circle C and let us ask similar questions. If C is distorted, as was our interval, by letting it slacken and bend but not fold onto itself or be broken violently, it is seen to retain its essential set structure. It retains the property, for example, of being severed by the removal of any two of its points whatever. Here again the property mentioned is characteristic for the type of curves which are topologically equivalent to the circle. In other words, if we define a *simple closed curve*

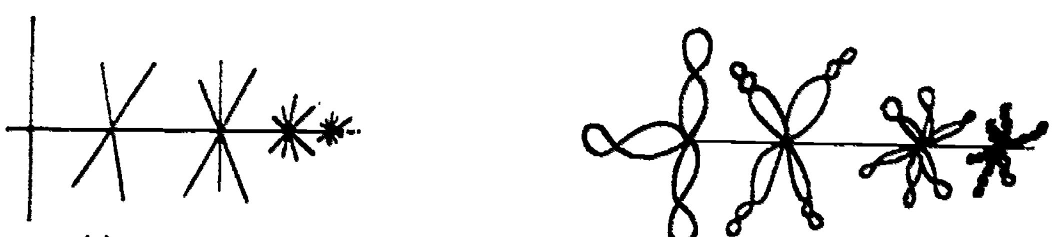
as a set which can be put in one-to-one and continuous correspondence with a circle, then *in order that a continuum C be a simple closed curve it is necessary and sufficient that C be disconnected by the omission of any two of its points*. Thus in Figure 2, (a) is not a simple closed curve since the removal of both *a* and *b* leaves the set connected. In Figure 3, (a) and (b) are simple closed curves but (c) is not a simple closed curve because the removal of *x* and *y* leaves the set connected.



(a) (b) (c)

FIG. 3

A curve which is made up of a finite number of simple arcs which overlap with each other only at end points of themselves is called a *graph* or a *linear graph*. A graph, then could be regarded as being constructed by putting together in any one of numerous ways a finite number of simple arcs so that no two of the arcs will overlap anywhere except possibly at an end point of both. All of the curves illus‹r›ated in Figs. 2 and 3 are graphs; and of course many more complicated structures could be made which would still be graphs. However, if a graph is in a plane it, like the simpler curves previously discussed, will have the classical property of 2-sidedness which does not belong to all curves.



(a) FIG. 4 (b)

Finally, we mention two further types of curves which in general are not graphs and yet whose structure is interesting and simple, namely the *dendrite or acyclic curve* and the *boundary curve*. A *dendrite* is a locally connected continuum which contains no simple closed curve. It may

contain infinitely many simple arcs [See Fig. 4 (a)]. In fact it may be impossible to express it as the sum even of countably many arcs, and yet it has the property that any two of its points are end points of one and only one arc in the curve. A *boundary curve* is a locally connected continuum which can be so imbedded in a plane that it will be the boundary of a connected region of the plane. Although it is true that every dendrite is a boundary curve, in general a boundary curve will contain one and may contain infinitely many simple closed curves [See Fig. 4 (b)]. However, it is interesting to note that no such curve could contain a cross bar on a simple closed curve. In other words, the most that any two simple closed curves can overlap is in a single point (point of "tangency"). Thus any boundary curve breaks up into so called cyclic elements which are either single points or simple closed curves, no two of these have more than one common point, and these fit together to make up the curve and give it a structure relative to these elements which is very similar to that of a dendrite [Compare Fig. 4 (a) with Fig. 4 (b)].

5. CONCLUSION. We have touched but a few of the many interesting aspects of the fundamental theory of curves. The subject has an extensive literature, particularly from the topological point of view, which the explorative reader will find fascinating as well as instructive. The field is a live one and it is currently receiving important contributions. Interesting and difficult problems remain unsolved. There is much to attract and repay the student who will expend the effort necessary to acquire a knowledge of these problems and to master the methods which have been devised for attacking them.

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